SOLUTION OF LIMIT LOAD PROBLEMS FOR A RIGID-PLASTIC ANISOTROPIC BODY

A. I. Chanyshev

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Using the results of [1, 2], we have examined and studied the equations of plane strain of a rigid-plastic anisotropic body. We have derived the characteristics and relations for the characteristics. As an example we have solved the Prandtl problem of the depression of a stamp in a rigid-plastic anisotropic medium. We have investigated the dependence of the limit load on the properties of the anisotropic body.

To simplify the discussion of the material we assume that in the fixed (x, y) coordinate system the anisotropic body obeys Hooke's law in the form*

$$\varepsilon_x = a_{11}\sigma_x - a_{12}\sigma_y,\ \varepsilon_y = -a_{12}\sigma_x + a_{22}\sigma_y,\ \varepsilon_{xy} = a_{33}\tau_{xy}, \tag{1}$$

where the a_{ij} are the elastic compliances ($a_{ij} > 0$), and $a_{11} \neq a_{22}$.

We determine the eigenvalues and characteristic tensors T_k ($T_k = ||t^k_{ij}||$) of the elastic compliance tensor [1]:

$$\lambda_1 = \frac{a_{11} + a_{22}}{2} + \sqrt{\left(\frac{a_{11} - a_{22}}{2}\right)^2 + a_{12}^2},\quad \lambda_2 = \frac{a_{11} + a_{22}}{2} - \sqrt{\left(\frac{a_{11} - a_{22}}{2}\right)^2 + a_{12}^2},\quad \lambda_3 = a_{33},$$

$$T_1:\quad t^1_x = \pm\frac{\sqrt{2}\,a_{12}}{\sqrt{a_{12}^2 + (a_{11} - \lambda_1)^2}},\quad t^1_y = \pm\frac{\sqrt{2}\,(a_{11} - \lambda_1)}{\sqrt{a_{12}^2 + (a_{11} - \lambda_1)^2}},\quad t^1_{xy} = 0,$$

$$T_2:\quad t^2_x = \pm\frac{\sqrt{2}\,a_{12}}{\sqrt{a_{12}^2 + (a_{11} - \lambda_2)^2}},\quad t^2_y = \pm\frac{\sqrt{2}\,(a_{11} - \lambda_2)}{\sqrt{a_{12}^2 + (a_{11} - \lambda_2)^2}},\quad t^2_{xy} = 0,$$

$$T_3:\quad t^3_x = t^3_y = 0,\quad t^3_{xy} = \pm 1.$$

It can be shown that

$$\frac{1}{2}\, t^k_{ij} t^l_{ij} = \delta_{kl}, \tag{2}$$

since $a_{11} - \lambda_1 = \lambda_2 - a_{22}$.†

It follows from (2) that the characteristic tensors T_k do not depend on the values of the stress tensor T_σ and the strain tensor T_ε. It is natural to assume that the orientations of the basic tensors T_k in tensor space are preserved even when plastic deformations appear [2]. This hypothesis will be important later in formulating the equations of a rigid-plastic anisotropic body.

We expand the stress tensor T_σ and the strain tensor T_ε in terms of the basis tensors T_k:

$$T_\sigma = S_k T_k,\ T_\varepsilon = \vartheta_k T_k,$$

where

*The assumption of Hooke's law in the form (1) does not limit the generality of the subsequent developments.

†Henceforth for definiteness we choose only the upper sign in Eqs. (2).

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$$S_1=\frac{1}{2}\left(\sigma_x t_x^1+\sigma_y t_y^1\right),\quad S_2=\frac{1}{2}\left(\sigma_x t_x^2+\sigma_y t_y^2\right),\quad S_3=\tau_{xy},$$
$$\text{Э}_1=\frac{1}{2}\left(\varepsilon_x t_x^1+\varepsilon_y t_y^1\right),\quad \text{Э}_2=\frac{1}{2}\left(\varepsilon_x t_x^2+\varepsilon_y t_y^2\right),\quad \text{Э}_3=\varepsilon_{xy}.$$

By virtue of the definition of the eigenvalues and the characteristic tensors T_k of the elastic compliance tensor we have

$$\text{Э}_k=\lambda_k S_k. \tag{3}$$

Different forms of the yield condition of an anisotropic medium and different configurations of a rigid-plastic anisotropic body are possible depending on the relations among the eigenvalues λ_k.

Let us consider the most typical situations.

A. Suppose $\lambda_1 \neq \lambda_1 \neq \lambda_3$, and that the yield condition is satisfied for deformation along the T_1 axis

$$|S_1|=k \quad \text{or} \quad \frac{1}{2}\left(\sigma_x t_x^1+\sigma_y t_y^1\right)=\pm k. \tag{4}$$

For a rigid-plastic material we also have equations for the strains of the form

$$\text{Э}_2=\text{Э}_3=0 \quad \text{or} \quad \frac{1}{2}\left(\varepsilon_x t_x^2+\varepsilon_y t_y^2\right)=0,\quad \varepsilon_{xy}=0. \tag{5}$$

In this way we obtain separate statically determinate problems for the stresses and displacements [3].

The characteristics of the system of equilibrium equations and the yield condition (4) have the form

$$\frac{dy}{dx}=\pm\sqrt{-\frac{t_x^1}{t_y^1}}. \tag{6}$$

The relations for the characteristics are $d\sigma_y = -d\tau dy/dx$. By considering system (5) for the displacements, we obtain characteristics coinciding with (6), and relations for the characteristics

$$du=-dv dy/dx.$$

B. Suppose $\lambda_1 = \lambda_3$, the yield condition has the form [2]

$$S_1^2+S_3^2=k^2 \quad \text{or} \quad \frac{1}{4}\left(\sigma_x t_x^1+\sigma_y t_y^1\right)^2+\tau_{xy}^2=k^2, \tag{7}$$

and the equations for the strains for the rigid-plastic body are such that

$$\text{Э}_3/S_3=\text{Э}_1/S_1,\quad \text{Э}_2=0 \quad \text{or} \quad \varepsilon_{xy}/\tau_{xy}=\left(\varepsilon_x t_x^1+\varepsilon_y t_y^1\right),\quad \varepsilon_x t_x^2+\varepsilon_y t_y^2=0. \tag{8}$$

Condition (7) expresses the fact that the stress vector on areas equally inclined to the principal axes of the tensor $S_1T_1 + S_3T_3$ remains constant; the first of conditions (8) means that the strain vector on the indicated areas is collinear with the stress vector; the second of conditions (8) indicates that the strain vector on areas equally inclined to the principal axes of the tensor T_2 varies elastically.

We note that for the usual Hooke's law ($a_{11} = a_{22}$) $t_x^1 = t_x^2 = t_y^2 = -t_y^1 = 1$, and $\lambda_1 = \lambda_3$, i.e., this is a special example of the case under consideration.

Supplementing the equilibrium equations by the yield condition (7), and introducing the variables σ and θ in the standard way:

$$\tau_{xy}=k\cos 2\theta,\quad \frac{1}{2}\left(\sigma_x t_x^1+\sigma_y t_y^1\right)=-k\sin 2\theta,\quad \frac{1}{2}\left(\sigma_x t_x^2+\sigma_y t_y^2\right)=\sigma,$$

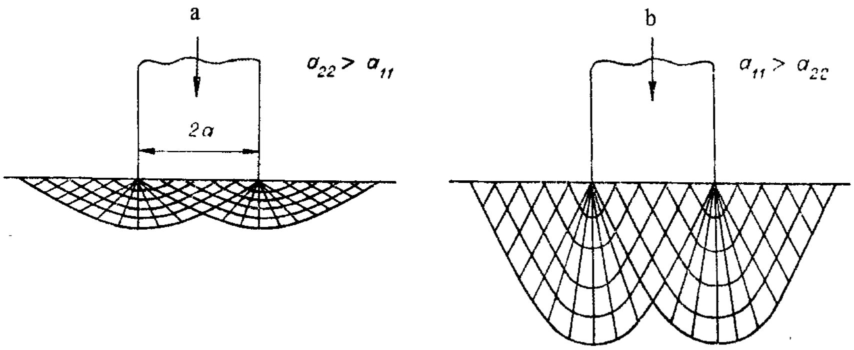

Fig. 1

we obtain a system of two nonlinear partial differential equations of the first order for the unknown functions σ(x, y) and θ(x, y). This system is of the hyperbolic type. The characteristic equation has the form

$$\mu^2 - 2\mu \frac{\operatorname{ctg} 2\theta}{t_y^1} + \frac{t_x^1}{t_y^1} = 0, \text{ where } \mu = \frac{dy}{dx}. \quad (9)$$

It follows from (9) that the characteristics generally do not intersect one another at right angles: if $a_{11} > a_{22}$, $t_x^1/t_y^1 < -1$, and if $a_{11} < a_{22}$, $t_x^1/t_y^1 > -1$.

We now present expressions of the characteristics and relations for the characteristics:

$$\mu_{1,2} = \frac{\operatorname{ctg} 2\theta \mp \sqrt{\operatorname{ctg}^2 2\theta - t_x^1 t_y^1}}{t_y^1}, \quad \sigma + \frac{k \sin 2\theta}{2}\left(\frac{t_y^2}{t_y^1} + \frac{t_x^2}{t_x^1}\right) \pm \frac{k\Delta}{2 t_x^1 t_y^1} E(2\theta, \varkappa) = \xi_{1,2},$$

where $\Delta = t_x^1 t_y^2 - t_y^1 t_x^2 = -\frac{2t_x^2}{t_y^1}$, $\varkappa = \sqrt{1 + t_x^1 t_y^1}$, and E is the elliptic integral of the second kind.

We now direct our attention to Eqs. (8). Expressing the strains in terms of the displacements, we obtain the equations

$$\frac{\partial u}{\partial x} t_x^2 + \frac{\partial v}{\partial y} t_y^2 = 0, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\left(\frac{\partial u}{\partial x} t_x^1 + \frac{\partial v}{\partial y} t_y^1\right) \operatorname{ctg} 2\theta,$$

which are also of the hyperbolic type with characteristics which coincide with those of the system of differential equations for the stresses. The relations for the characteristics have the form

$$du = -dv\,dy/dx.$$

We note that in the case under consideration, just as in [4], simple stressed states occur.

To illustrate the application of the relations derived, we solve the Prandtl problem of the impression of a stamp in a rigid-plastic anisotropic medium. We assume that the plastic medium is bounded by a plane, and that there is no friction on the surface of contact. In the limiting state the stamp is moved downward (Fig. 1). It is required to determine the limit load corresponding to the onset of plastic flow. We present a solution similar to Prandtl's. The distribution of slip lines may be of two types, depending on the relation between the elastic compliances a_{11} and a_{22} (Fig. 1a, b): In the first case ($a_{11} > a_{22}$) the characteristics approach the free surface at an angle less than 45° to the x axis; in the second case ($a_{11} > a_{22}$) this angle is more than 45°. The limit load in both cases is calculated from the formula

$$P_* = -\frac{4ak}{t_y^1}\left(1 + E\left(\frac{\pi}{2}, \varkappa\right)\right).$$

It is easy to show that the limit load for fixed k is a function of the parameter $\alpha = 2a_{12}/(a_{11} - a_{22})$. The limit load reaches extreme values as $\alpha \to \pm 0$: as $\alpha \to +0$, the limit load increases without bound (Fig. 1b); as $\alpha \to -0$ the limit load approaches $4\sqrt{2a}k$ (Fig. 1a); as $\alpha \to \pm\infty$ we obtain the loading obtained by Prandtl.

C. Suppose $\lambda_1 = \lambda_2 = \lambda_3$. In this case $a_{11} = a_{22} = a_{33}$, $a_{12} = 0$, and the orthonormal tensor basis T_k is not uniquely determined by Eqs. (2). The following can be taken as the orthonormal tensor basis:

$$T_1:\ t_x^1 = -t_y^1 = 1,\ t_{xy}^1 = 0,\quad T_2:\ t_x^2 = t_y^2 = 1,\ t_{xy}^2 = 0,\quad T_3:\ t_x^3 = t_y^3 = 0,\ t_{xy}^3 = 1.$$

In this case the yield condition will have the form

$$\frac{1}{4}(\sigma_x + \sigma_y)^2 + \frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2 = k^2.$$

We introduce the variables θ and α in the following way:

$$(\sigma_x + \sigma_y)/2 = k \sin\alpha,\ (\sigma_x - \sigma_y)/2 = -k \sin 2\theta \cos\alpha,\ \tau_{xy} = k \cos 2\theta \cos\alpha. \tag{10}$$

Substituting (10) into the equilibrium equations, we obtain a system of two nonlinear partial differential equations of the first order for the unknown equations θ(x, y) and α(x, y). This system is of the hyperbolic type for $-\pi/4 \leqslant \alpha \leqslant \pi/4$. Its characteristics and the relations for the characteristics are the following:

$$\mu_{1,2} = \frac{-\cos 2\theta \cos\alpha \pm \sqrt{\cos 2\alpha}}{\sin\alpha + \sin 2\theta \cos\alpha},\quad 2\theta = \pm(-u + \sqrt{2}\,\mathrm{arctg}(\sqrt{2}\,\mathrm{tg}\,u)) + \mathrm{const},$$
$$\text{where } u = \arcsin(\mathrm{tg}\,\alpha).$$

Expressing the collinearity condition for the strain and stress vectors on surfaces which are equally inclined to the principal axes of the tensor T_σ, we have

$$\frac{\varepsilon_x + \varepsilon_y}{2\varepsilon_{xy}} = \frac{\sigma_x + \sigma_y}{2\tau_{xy}},\quad \frac{\varepsilon_x - \varepsilon_y}{2\varepsilon_{xy}} = \frac{\sigma_x - \sigma_y}{2\tau_{xy}}.$$

Substituting the expressions for the strains in terms of the displacements, we obtain a system of two differential equations for the displacements. This system is also of the hyperbolic type. Its characteristics coincide with those of the system of differential equations for the stresses, and the relations for the characteristics have the same form as in the cases analyzed earlier:

$$du = -dv dy/dx.$$

The examples presented show how diverse the plastic properties of anisotropic media can be. These properties are dictated by the structural features of the medium, which, in the first approximation, are determined by the elastic compliance matrix.

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